University of Baghdad
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Unsteady Flow of Non-Newtonian Fluid in a Curved Pipe with Rectangular Cross-Section

A Thesis
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Degree of Master of Science
in Mathematics.

By

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بِسمِ اللهِ
الرَحمَنِ الرَحِيمِ

"أَلَم تَرَ كَيفَ ضَرَبَ ٱللهُ مَثَلاً كَلِمَةً طَيِبَةً كَشَجَرَةٍ طَيِبَةٍ

أَصلُهَا ثَابِتٌ وَفَرعُهَا فِي السَمَآءِ"

صَدَقَ ألله العَظِيمْ
إبراهيم 24

To...

My Family

شــكر وتقديــــر

بسم الله الرحمن الرحيم , الحمد لله الذي ابتدأني بالنعمة فمنَ علي بالحياة واوجدني من العدم ثم رزقني ورباني الى ان وصلت ما وصلت اليه ...
واصلي واسلم على رسوله الذي به اهتديت الى الدين الحق وهو الذي حثني على طلب العلم محمد بن عبدالله وعلى اله الطاهرين وصحبه أجمعين.
واقدم شكري وامتناني الى نبع المودة والرحمة والعطف والديَ الكريمين ... الى من تعاهدا هذه الشجرة الضعيفة بالسقي والرعاية الى ان انشد عودها واينعت ثمارها والى الان لم يتركاها ... فلا حرمني الله من هذا النبع المبارك.
وشكري وتقديري الى اخوتي الاعزاء والى عائلتي الكريمة.
واقدم شكري و ثنائي واعتزازي الى رئيس واساتذة قسم علوم الرياضيات ... وان كانت كلمات الشكر عاجزة عن تادية حقهم, ولكني تيمنت بقول الشاعر:

وان اولاك ذو فضل جميلا

فكن بالشكر منطلق اللسان

وهذا قلمي ولساني ينطلقان بالشكر لاساتذتي الذين اعانوني كثيرا, فيسروا لي العسير وقربوا لي البعيد حتى تمكنت بعون الله من انجاز هذا العمل.
ولا انسى مواقف حضرة الدكتور الفاضل عبد الرحمن حميد رئيس قسم الرياضيات وما قدم لي من تسهيلات, فجزاه الله خير جزاء المحسنين...
وكذلك لا انسى جهود استاذي الفاضل الذي اشرف على اطروحتي الدكتور احمد مولود فشكر الله سعيه ولا انساني فضله ما بقيت.
واخص بالشكر والتقدير الاستاذ الدكتور رياض شاكر نعوم والدكتورة بثينة على ما ابدياه لي من عون والذي كانا سبباً في أستمرار دراستي وأسال الله ان يحفظهم من كل سوء.
واشكر كل اصدقائي الذين مدوا لي يد العون وساعدوني كثيرا ... ولله در من قال :

دعوى الاخاء على الرخاء كثيرة

بل في الشدائد تعرف الاخوان

وفي الختام لايسعني الا ان اكرر شكري وتقديري لكل من اعانني على انجاز هذا العمل والحمد لله اولا واخرا.

محمد علي مراد

Certificate

I certify that this thesis **"Unsteady Flow of Non-Newtonian Fluid in a Curved Pipe with Rectangular Cross-Section"** was prepared under my supervision at the University of Baghdad as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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In view of the available recommendations, I forward this thesis for debate by the examining committee.

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Committee report

We certify that we have read this thesis **"Unsteady Flow of Non- Newtonian Fluid in a Curved Pipe with Rectangular Cross-Section"** and, as an examining committee examined the student in its context and that in our opinion it is adequate for the partial fulfillment of the requirements for the degree of Master of Science in Mathematics with, ( ) standing .

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Abstract

This thesis is concerned with the study of unsteady flow of non-Newtonian, viscous, incompressible fluid in a curved pipe with rectangular cross-section, under the action of pressure gradient. Consideration is given to two cases, wide and longitudinal rectangular. An orthogonal coordinates system has been used to describe the fluid motion for each case and it is found that the motion equations are controlled by three parameters namely; Dean number, non-Newtonian parameter and frequency parameter.

For each case, solution for the secondary flow and the axial velocity are drived as perturbation over straight pipe. Firstly the expansion was in terms of Dean number and secondly in terms of frequency parameter. Perturbation equations are solved by using a variational method namely, Galerkin's method after eliminating the dependence on time for each case. The solutions have been developed in Cartesian coordinates for harmonic and biharmonic equations. In This study we covered the steady state for both cases under consideration.

QBASIC language is used to make the numerical computations of these solutions, while the MATLAB package is used to draw the figures of stream function and axial velocity. Our study is ended with studying the effect of the non –dimensional parameters mentioned above on the secondary flow, the axial velocity and the flow in the central plane.

List of Symbols

Symbols	Descriptions
T_{ik}	Stress Tensor
e_{ik}	Rate of Strain
υ	Kinematic Viscosity
ξ	Normal Stress
R_e	Reynolds Number
L	Dean Number
β	Non-Newtonian parameter
K	Frequency Parameter
P	Pressure
ρ	Density
t	Time
τ	Dimensionless Time
ψ	Dimensional Stream Function
f	Dimensionless Stream Function
U, V, W	Dimensional Velocity in the x, θ, z Directions
u, v, w	Dimensionless Velocity in the x, θ, z Directions
d	Rectangular Length
h	Rectangular Height
∇^2	Laplacian Operator
∇^4	Biharmonic Operator
$\breve{T}$	Unite Tangent Vector
α	Angular Frequency
$c(s)$	Curvature
R	Radius of the Curved Pipe

Contents

Introduction

Fluid mechanics is that branch of applied mathematics which deals with behavior of fluid at rest and in motion. Fluid is that state of matter which is capable of changing shape and is capable of flowing. Each fluid characterized by an equation that relates stress to rate of strain, known as "state equation ". Fluid may be classified as "viscous" or "perfect" according to whether the fluid is capable of exerting shear stress or not. Viscous fluid is called Newtonian if the relation between stress and rate of strain, in state equation, is linear, otherwise it is called non-Newtonian fluid.

Viscous flow through straight ducts of various cross-section forms is well understood. The flow in a gently curved duct may be considered as a modification of straight axial flow in which the effect of centrifugal forces must be considered.

Dean, (1927),[7] is the first researcher who worked in flow analysis of Newtonian fluids in curved pipes. He introduced a toroidal coordinate system to show that the relation between pressure gradient and the rate of flow through a curved pipe with circular cross-section of incompressible Newtonian fluid is dependant on the curvature. In that paper he could not show this dependence but he did it in his second paper (1928),[8] where he modified his analysis by including higher order terms to be able to show that the rate of flow is slightly reduced by curvature.

Dean and Harst (1957),[10] obtained an approximate solution of Newtonian fluid flow in a curved pipe with rectangular cross-section assuming that the secondary motion is a uniformly stream from inner to outer bend. They modeled the equations of motion by using cylindrical coordinates. This assumption enabled them to obtain Bessel's function solution. They argued that the secondary motion decreases the rate of flow produced by a given pressure gradient and causes an outward movement at the region where the prime motion is the greatest.

In his paper Jones (1960),[17] makes a theoretical analysis of the flow of incompressible Non-Newtonian viscous liquid in a curved pipe with circular cross-section keeping only the first order terms. He shows that the secondary motion consists of two symmetrical vortices and the distance of the stream lines from the central plane decreases as the Non-Newtonian parameter increases.

Past work on fully developed flow in a curved square duct includes numerical studies by Mori, Uchida & Ukon (1971),[23] who obtained a numerical solution by using boundary-layer approximation (valid for large Dean numbers); Cheng. Lin & Ou (1976),[7], Ghia & Shokhey (1977),[14] and Joseph Smith & Adler (1975),[18] who obtained solutions which predicted the existence of a weak second vortex pair near the outer wall above a certain value of the Dean number. This second vortex pair was found to rotate in the opposite manner to the primary vortex pair. Cheng *et al* (1976),[7] predicted the onset of second vortex pair to occur when a Dean number is >150.

Ghia & Sokhey,[14] predict in it to occur above a Dean number of 143 while the calculations of Joseph *et al*,[18], give a threshold Dean number of 152 since the curvature ratio (whose effect is embedded in the Dean number) may itself play an important role for highly curved ducts. The suitability of the Dean number as the sole parameter to characterize the onset of the second vortex pair is unclear.

For curved rectangular ducts Cheng *et al* (1976),[7] performed calculations for duct aspect ratio (defined as the ratio of height H to the width B) of 0.5, 2 and 5 for the range of the Dean number 15.9 to 312.7 at curvature ratios of 100 and 30. They reported that for an aspect ratio of 0.5 at L=176 there were no additional vortices and at L=200 there was a pair of very weak vortices close to the outer wall. In addition they found that for an aspect ratio of 5 a pair of secondary vortices appeared at a rather low Dean number of 76 and the eye of the primary vortex moved toward the upper and the lower walls with the increase of Dean number.

Winter, K. H. (1987),[34] considers the bifurcation of secondary solutions for fully developed laminar flow in curved rectangular ducts. The study is based on finite-element analysis and shows the existence of the multiple solutions arising from the non-linear equations for the range of aspect ratio from 0.8 to 1.6.

Ravi Sankar, Nandakumar & Masliyah (1988),[24] consider the related problems of developing flow in curved ducts. They have shown that for a range of curvature ratios and Dean numbers the flow develop into previously known two-and four- cell patterns based on fully three-dimensional calculations using the parabolized form of the Navier-Stokes equations. They have also shown that for loosely coiled ducts (of curvature ratio of 100) outside a narrow range of Dean number the solution exhibits sustained oscillations in the axial direction and that no stable steady solutions could be predicted.

Thangam and Hur,(1990),[30] show that the secondary flow of incompressible viscous fluid in a curved duct is studied by using a finite-volume method. It is shown that as Dean number is increased the secondary flow structure evolves into a double vortex pair for low -aspect- ratio duct and roll cell for duct of high aspect ratio. They found that for ducts of high curvature the onset of transition from single vortex pair to a double vortex pair or roll cells depends on the Dean number and the curvature ratio while for ducts of small curvature the onset can be characterized by Dean number alone.

Jing-Wu Wang and Andrews, in (1995),[16] use a non-orthogonal coordinate system to study the effect of the pitch ratio and curvature on the velocity distribution of fully developed laminar flow of an incompressible fluid in a helical duct with rectangular cross-section. They used a numerical method to solve the motion equations, they find that the pitch ratio affects the pattern of the secondary flow, two-vortex become a single vortex if the pitch ratio is greater than 10 and for a certain level there will be four vortexes to appear on the plan of the cross-section.

Yakhot A., et al (1999),[35] studied a pulsating laminar flow of a viscous, incompressible liquid in a rectangular duct . The motion is induced under an imposed pulsating pressure difference. The problem is solved numerically. Difference flow regimes are characterized by non- dimensional parameters based on the frequency of the imposed pressure gradient oscillation and the width of the duct. The influence of the aspect ratio of the rectangular duct and the pulsating pressure gradient frequency on the phase lag, the amplitude of the induced oscillating velocity, and the wall shear were analyzed.

Abdul-Hadi A. M.(2000),[1] studied the unsteady flow of incompressible non-Newtonian fluid in a curved pipe with a square cross-section. He used a Galerkin method which is a variational method to solve the equations of Navier-Stokes. He shows that a secondary motion depends on three dimensional parameters namely Dean number, non-Newtonian and frequency parameters, also he studied the effect of these three parameters on the secondary flow, axial velocity and some other relation.

AL-Musawy A. Z. H. (2004),[2] studied the flow of non-Newtonian fluid in a curved duct with varying aspect ratio. In his computation he used a Galerkin method and finite-difference to solve the equations of Navier-stokes. He shows that a secondary motion depends on two dimensional parameters, also he studied the effect of non-Newtonian and aspect ratio parameters on the secondary flow and axial velocity.

Our work will be generalized to chapter tow of Abdul-Hadi A. M. work. This thesis contains four chapters:-

Chapter one devoted to study some of fluid properties and basic concepts.

Chapter two deals with unsteady flow of viscous, incompressible, non-Newtonian fluid in curved pipe with rectangular cross-section. An orthogonal coordinates system has been formed to describe the fluid motion. In this chapter we are going to study two cases, wide rectangular and longitudinal rectangular. In each case the motion equations are controlled by three parameters namely, Dean number, non-Newtonian parameter and frequency parameter. In each case, solution of the secondary flow and the axial velocity are described by perturbations over straight pipe appearing the Dean number.

Chapter three contains solutions of the problem for case1 and case2. These solutions are firstly expanded in terms of Dean number (chapter two) and secondly in terms of frequency parameter. Perturbations equations are solved by Galerkin method after eliminating the dependence on time.

In chapter four we study the effect of the parameters mentioned above on the flow in the central plane, the secondary motion and the axial velocity for each case. This chapter ended with studying a comparison between case1 and case2.

CHAPTER ONE

Some Definitions and Basic Concepts

Introduction

The study of fluid dynamics is of closed link with the physical properties of fluids such as density, viscosity, pressure …etc. As an introduction to some of the issues in the mechanics of fluid, this chapter will include a preliminary discussion of a few such properties fluids flows.

1.1 Density

The density of a fluid, denoted by ρ, in unit of Kg/m^3 is defined as the mass per unit volume of the fluid,

$$\rho = \frac{m}{V} \qquad \text{…. (1-1)}$$

where m is the mass and V the volume. According to this property, fluids can be classified into compressible and incompressible. When the density is constant, the fluid is known as incompressible but when it changes with time, the fluid is known as compressible. [28]

1.2 Viscosity

A viscosity of fluid is that characteristic of real fluid which exhibits a certain resistance to change of form. Some of viscous fluids "Newtonian fluids" obeys the linear relationship given by Newton's law of viscosity.

$$T = \eta \frac{du}{dy} \qquad \text{…. (1-2)}$$

where T is the shear stress (force per unit area), $\frac{du}{dy}$ is called as velocity gradient and η is the coefficient of dynamic viscosity or simply called viscosity. [28]

1.2.1 Coefficient of Dynamic Viscosity:

The viscosity is defined as the tangential force required per unit area to sustain a unit velocity gradient. [28]

1.2.2 Kinematic Viscosity:

Kinematic viscosity, denoted by υ, in units of m^2/s is defined as the ratio of dynamic viscosity to mass density. [28]

$$\upsilon = \frac{\eta}{\rho} \qquad \text{.... (1-3)}$$

1.3 Pressure

Pressure, denoted by *P*, in units Kg/m.s^^{2} is defined as the local normal force per unit area,

$$P = \frac{F_n}{A} \qquad \text{.... (1-4)}$$

where F_n is the normal forces to surface with area *A*. [28]

1.4 Fluid Flow

Historically, flow phenomena have been studied by the most famous thinkers of antiquity and, more recently, by the most notable mathematicians and experimenters. In internal flow through pipes, channels …etc, the flow is established and sustained by to overcome the resistance of flow.

It is possible –and useful-to classify the type of flow which is being examined into small number of groups. If we look at a fluid flowing under normal circumstances –a river for example –the conditions at one point will vary from those at another point (e.g. different velocity) we have unsteady flow.

Under some circumstances the flow will not be as changeable as this. In what follow, we are going to define the terms describing the states which are used to classify flow. [3]

1.4.1 Uniform Flow:

If the flow velocity is same in magnitude and direction at every point in the fluid, then the flow is said to be uniform. [3], [28], [29]

1.4.2 Non – Uniform Flow :

If at a given instant, the velocity is not same at every point then the flow is non-uniform flow. [3], [28], [29]

1.4.3 Steady Flow:

A steady flow is one in which one of the following (velocity, pressure and cross- section) may differ from point to point but do not change with time. [3], [28], [29]

1.4.4 Unsteady Flow:

If at any point in the fluid, the condition change with time, then the flow is described as unsteady. [3], [28], [29]

1.4.5 Laminar Flow:

If the fluid particles move along smooth, regular paths, then the flow is called laminar flow. [28].

1.4.6 Turbulent Flow:

If the fluid particles move randomly, then the flow is called turbulent flow. [28]

1.5 Reynolds' Number

The dimensionless expression $\rho v d / \eta$ where ρ, v, d and η are density, mean velocity, diameter and dynamic viscosity respectively, is called Reynolds number .The value of Reynolds number help us to predict the change in flow type. If its value less than about 2000 then the flow is laminar, if greater than 4000 then the flow is turbulent and in between these then in the transition state from laminar to turbulent. [3], [28], [29]

1.6 Continuity Equation

The continuity equation simply expresses the law of conservation of mass (the mass per unit time entering the tube must be flow out at the same rate) in mathematical form. [3], [28], [29]

1.7 Motion Equations

The motion equations are non-linear (or linear sometime) partial differential equations which expressed the Newton's second law in mathematical form. Thus the motion equations can be eveloped from consideration of the force acting on a small element of the fluid, including the shear stresses generated by fluid motion and viscosity. [28], [29]

1.8 Stream Function

Let A be a fixed point in the plane of motion, and ABP, ACP are two curves joining A to an arbitrary point P, Fig.(1).

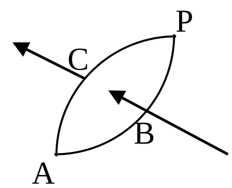


Fig.(1), Stream function

According to the continuity equation the flux through ABP is equal to the flux through ACP. If we denote the flux by the function ψ, then ψ depends on the position of P and time, i.e. $\psi(x, y, t)$.The function ψ is called stream function. [28]

Fig.(2), illustrate the relation between the stream function $\psi(x, z, 0)$ and the velocity field.

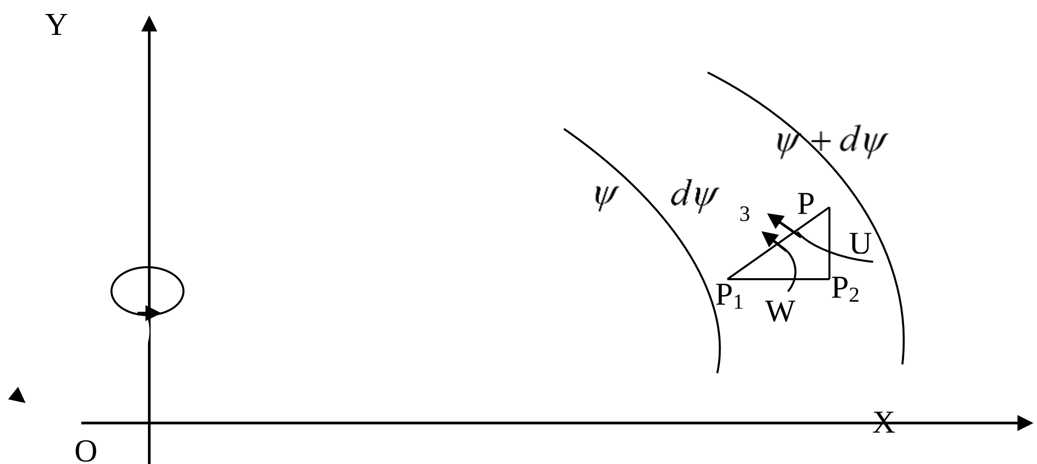


Fig. (2),Stream function and velocity field

From the continuity equation we have

The flux through $\overline{P_1P_3}$ =flux through $\overline{P_1P_2}$ +flux through $\overline{P_2P_3}$

$$d\psi = -Udy + Wdx \qquad \text{…. (1-5)}$$

since $\psi = \psi(x, y)$,then by chain rule we have

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \qquad \text{…. (1-6)}$$

from (1-5) and (1-6) we get

$$U = -\frac{\partial \psi}{\partial y} \quad ,W = \frac{\partial \psi}{\partial x} \qquad \text{…. (1-7)}$$

1.8.1 Streamline:

A streamline is an imaginary line drawn through the flow field such that the tangent at any point is in the direction of the velocity vector. [3], [28], [29]

1.8.2 Theorem:

A steam function is constant along a streamline. [28]

1.9 Stress and Strain

Fluid particles in motion deform, and therefore, can conform to complex geometries and shapes. In practical terms, deformation is represents the different ways in which particles can change shape or position under the influence of external forces. This deformation defined as “strain”.The external forces when transmitted in the fluid particles develop to internal forces. For convenience, internal forces are expressed in terms of stresses denoted by *T*, and stress is defined as the force per unit area along which the force acting on. Consequently, stress and force are equivalent concept.

In Fig.(3) we noted that stresses are distinguished as normal and tangential.

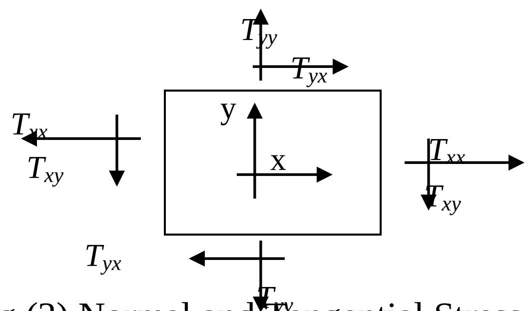


Fig.(3),Normal and Tangential Stresses

the subscripts $_{xy}$ or $_{yx}$ on the tangential stresses indicate respectively the face and direction the stresses are applied to. To guarantee static equilibrium for the free body diagram in Fig.(3) and, therefore, to ensure that $\sum F = 0$ and $\sum M = 0$, where F and M respectively the force and moment vectors, we must have $T_{xy}=T_{yx}$.

By including the third direction, the stress state at a particular point in a three-dimensional flow is given by the tensor:-

$$T=\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

Again, for static equilibrium, $T_{xy} = T_{yx}, T_{xz} = T_{zx}, T_{yz} = T_{zy}$, which makes the stress tensor symmetric (i.e. off-diagonal terms are equal). [3]

1.10 Curvature

If $\breve{T}$ is the unit tangent vector of a smooth curve, the curvature function of the curve is

$$c(s) = \left| \frac{d\breve{T}}{ds} \right| \qquad \text{.... (1-8).}$$

If $\left| \frac{d\breve{T}}{ds} \right|$ is large, the curvature at p is large, if $\left| \frac{d\breve{T}}{ds} \right|$ is close to zero, the curvature at p is smaller. Fig.(4), describe the curvature. [3]

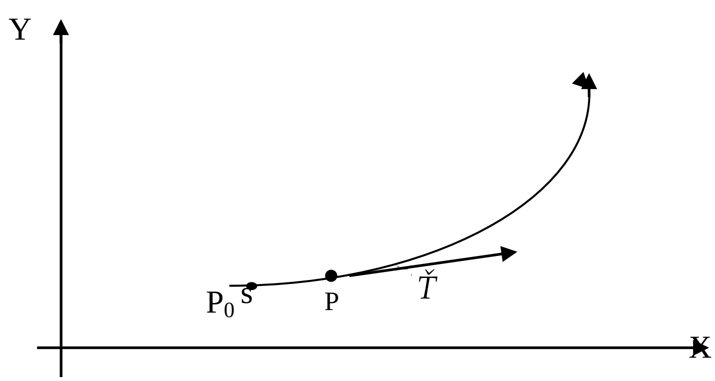


Fig.(4),The Curvature

1.11 Dimensional Analysis

Any physical phenomena can be described by certain quantitative properties e.g. length, velocity, area, volume…etc. These are known as dimensions. Of course dimensions are of no use without a magnitude being attached. We must know more than that something has a length. It must also have a standardized unit-such as a meter, a foot …etc.

Dimensions are properties which can be measured. Units are the standard elements we use to quantify these dimensions. In dimensional analysis we are only concerned with the nature of the dimension i.e. it's quantity. Thus, the dimensional analysis is a method to describe natural phenomena by a dimensionally correct equation among certain variables which affect the phenomena. There are several methods in the dimensional analysis; one of these methods is described in the following subsection. [3], [28]

1.11.1 Scaling and Order-of-Magnitude Analysis:

This method consists of two steps

Step1:

Scale the flow variables using quantities characteristic of the flow. For example in the flow in pipes the choice for characterizing length and velocity scales are respectively the diameter D of the pipe and free stream velocity V, then $\bar{u}=\frac{u}{V}$ and $\bar{v}=\frac{v}{V}$ are dimensionless quantities.

Step2:

Extend the scaling procedure to all terms in the governing equation.

The above procedure leads naturally into the non-dimensionalization of the continuity and motion equations.

CHAPTER TWO

Formulation of the Problem

Introduction

The problem under consideration is an unsteady flow of viscose, incompressible, non-Newtonian fluid in a curved pipe with rectangular cross-section. To describe the flow a cylindrical coordinates, orthogonal coordinates, are used. It is shown that the dimensionless motion equations are controlled by three parameters namely Dean number D, non-Newtonian parameter β and the frequency parameter k. The linearization of motion equations has been done by using a series solution of ascending power of Dean number.

2.1 A Mathematical Consideration

Unsteady flow of non-Newtonian fluid in a curved pipe is considered. The non-Newtonian fluid is characterized by equation of state of the form:

$$T_{ik} = 2\eta\, e_{ik} + 4\xi\, e_{ij} e_{jk} . \quad {}_{i=1,2,3} \quad {}_{j=1,2,3} \;\; {}_{k=1,2,3} . \qquad \ldots(2\text{-}1)$$

where $T_{ik}, e_{ik}, \eta \; and \; \xi$ are the stress, rate of strain, viscosity coefficient and normal stress respectively. [26]

Fig.(5), illustrates the coordinates system that has been used. OZ is the axis of the circle formed by the wall of the pipe. C is the center of the section of the pipe by a plane through OZ making an angle θ with a fixed axial plane. CO is the perpendicular drawn from C upon OZ and is of length R .The plane through O perpendicular to OZ and the line traced out by C will be called the central plane and the center line of the pipe respectively. Cartesian coordinates x and z are drawn in the section of the pipe, where x is parallel to OC and z parallel to OZ. The position of any point Q is then specified by cylindrical coordinate (x,θ, z), $-d \le x \le d$ and $-h \le z \le h$ where d and h are the length and height of the cross-section respectively. The Cartesian system (X,Y,Z) is related to the coordinate system in the cross-section by the relations

$$X = (R + x)\mathrm{Cos}(\theta), \quad Y = (R + x)\mathrm{Sin}(\theta), \quad Z = z \qquad \ldots. \ (2\text{-}2)$$

where $o \le \theta \le 2\pi$.

Two cases will be examined for convenient length:
case1, when $d = 3, h = 2$, see fig.(5) and case2, when $d = 2, h = 3$, see fig.(6).

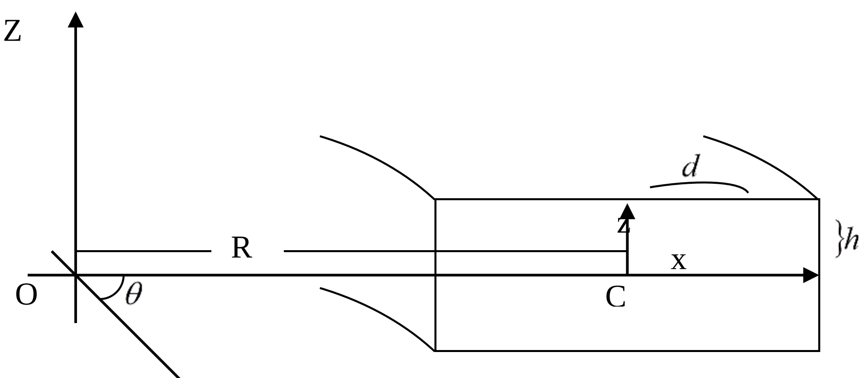


Fig.(5), Coordinates system

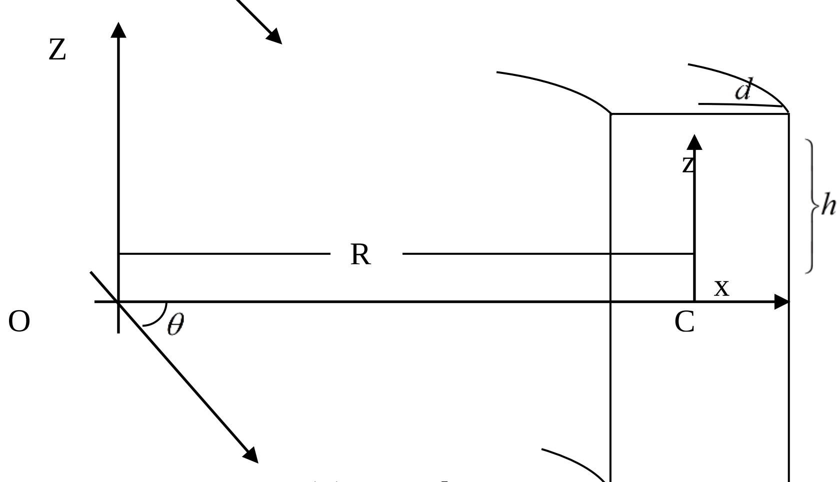


Fig.(6),Coordinates system

The line element is given by

$$(ds)^2 = (dx)^2 + (R+x)^2 (d\theta)^2 + (dz)^2 \qquad .... (2\text{-}3)$$

It is clear from (2-3) that the coordinate system (X,θ,Z) is orthogonal system. So it is possible to use the curvilinear coordinate to write down the continuity equation and motion equations

The line element in curvilinear coordinate is given by, [27]

$$(ds)^2 = (\frac{dx}{h_1})^2 + (\frac{dy}{h_2})^2 + (\frac{dz}{h_3})^2 \qquad .... (2\text{-}4)$$

where $\frac{1}{h_1}, \frac{1}{h_2}$ *and* $\frac{1}{h_3}$ are the coefficient of dx, dy *and* dz respectively.

Then in comparison equation (2-3) with equation (2-4) we have

$$h_1 = 1,\ h_2 = \frac{1}{R+x},\ h_3 = 1.$$

<u>2.2 The Curvilinear Coordinates of the Stress and Rate of Strain Components</u>

Let (U,V,W) be the velocity component in the direction coordinates (x,θ,z). Then physical components of the rate of strain are, [27]

$$\left.\begin{aligned}
&e_{xx} = \frac{\partial U}{\partial x} \quad , e_{\theta\theta} = \frac{1}{R+x}\left(\frac{\partial V}{\partial \theta} + U\right) \ , e_{zz} = \frac{\partial W}{\partial z}\ , \\
&e_{\theta z} = e_{z\theta} = \frac{1}{2}\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right), \\
&e_{zx} = e_{xz} = \frac{1}{2}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right), \\
&e_{x\theta} = e_{\theta x} = \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right)
\end{aligned}\right\} \qquad .... (2\text{-}5)$$

By using equation of (2-1) and (2-5) the physical components stress can be written as

$$
\left.
\begin{aligned}
T_{xx} &= 2\eta\frac{\partial U}{\partial x} + 4\xi\left(\frac{\partial U}{\partial x}\right)^2 + \xi\left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right)^2 \\
&\quad + \xi\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)^2 \\
T_{\theta\theta} &= 2\eta\frac{1}{R+x}\left(\frac{\partial V}{\partial \theta} + U\right) + \xi\left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right)^2 \\
&\quad + 4\xi\frac{1}{(R+x)^2}\left(\frac{\partial V}{\partial \theta} + U\right)^2 + \xi\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right)^2 \\
T_{zz} &= 2\eta\frac{\partial W}{\partial z} + \xi\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)^2 + \xi\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right)^2 \\
&\quad + 4\xi\left(\frac{\partial W}{\partial z}\right)^2 \\
T_{z\theta} &= T_{\theta z} = \eta\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right) + \xi\left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right) \\
&\quad \left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) + 2\xi\frac{1}{R+x}\left(\frac{\partial V}{\partial \theta} + U\right)\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right) \\
&\quad + 2\xi\left(\frac{\partial W}{\partial z}\right)\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right) \\
T_{xz} &= T_{zx} = \eta\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) + 2\xi\left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) + \\
&\quad \xi\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right) * \\
&\quad \left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right) + 2\xi\left(\frac{\partial W}{\partial z}\right)\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) \\
T_{\theta x} &= T_{x\theta} = \eta\left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right) + 2\xi\left(\frac{\partial U}{\partial x}\right) * \\
&\quad \left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right) + 2\xi\left(\frac{\partial V}{\partial x} - \frac{V}{R+x} + \frac{1}{R+x}\frac{\partial U}{\partial \theta}\right) \\
&\quad * \frac{1}{R+x}\left(\frac{\partial V}{\partial \theta} + U\right) + \xi\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)\left(\frac{1}{R+x}\frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z}\right)
\end{aligned}
\right\} \cdots (2\text{-}6)
$$

2.3 The Continuity and Motion Equations

The continuity and motion equations for non-Newtonian fluid are (28),

$$\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}=0 \qquad \text{…. (2-7)}$$

$$\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+W\frac{\partial U}{\partial z}-\frac{V^2}{R}\right)=-\frac{\partial P}{\partial x}+\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xz}}{\partial z}-\frac{T_{\theta\theta}}{R} \qquad \text{..(2-8)}$$

$$\rho\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+W\frac{\partial V}{\partial z}\right)=-\frac{1}{R}\frac{\partial P}{\partial \theta}+\frac{\partial T_{x\theta}}{\partial x}+\frac{\partial T_{\theta z}}{\partial z} \qquad \text{…. (2-9)}$$

$$\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+W\frac{\partial W}{\partial z}\right)=-\frac{\partial P}{\partial z}+\frac{\partial T_{xz}}{\partial x}+\frac{\partial T_{zz}}{\partial z} \qquad \text{…. (2-10)}$$

In equations (2-7)-(2-10), we assume that the fluid is incompressible (ρ=constant) and the velocity component (U, V, W) are independent of θ but the pressure p is not.

Substituting equations (2-6) in (2-8)-(2-10) gives

$$
\begin{aligned}
\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+W\frac{\partial U}{\partial z}-\frac{V^{2}}{R}\right)&=-\frac{\partial P}{\partial x}+2\eta\frac{\partial^{2}U}{\partial x^{2}}+8\xi\frac{\partial U}{\partial x}\frac{\partial^{2}U}{\partial x^{2}}\\
&+2\xi\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)+\eta\left(\frac{\partial^{2}U}{\partial x\partial z}+\frac{\partial^{2}W}{\partial x^{2}}\right)+2\xi\frac{\partial^{2}U}{\partial x\partial z}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)\\
&+2\xi\frac{\partial U}{\partial x}\left(\frac{\partial^{2}U}{\partial z^{2}}+\frac{\partial^{2}W}{\partial x\partial z}\right)+2\xi\frac{\partial^{2}W}{\partial z^{2}}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)+2\xi\frac{\partial W}{\partial z}\\
&\left(\frac{\partial^{2}U}{\partial z^{2}}+\frac{\partial^{2}W}{\partial x\partial z}\right)-2\frac{\eta}{R^{2}}U-\frac{\xi}{R}\frac{\partial^{2}V}{\partial x^{2}}-4\frac{\xi}{R^{3}}U^{2}-\frac{\xi}{R}\frac{\partial^{2}V}{\partial z^{2}}\qquad \text{…. (2-11)}
\end{aligned}
$$

$$
\begin{aligned}
\rho\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+W\frac{\partial V}{\partial z}\right)&=-\frac{1}{R}\frac{\partial P}{\partial\theta}+\eta\frac{\partial^{2}V}{\partial x^{2}}+2\xi\frac{\partial^{2}U}{\partial x^{2}}\frac{\partial V}{\partial x}\\
&+2\xi\frac{\partial U}{\partial x}\frac{\partial^{2}V}{\partial x^{2}}+\xi\frac{\partial V}{\partial z}\left(\frac{\partial^{2}U}{\partial x\partial z}+\frac{\partial^{2}W}{\partial x^{2}}\right)\quad 2\xi\frac{\partial^{2}V}{\partial x\partial z}\\
&\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)+\eta\frac{\partial^{2}V}{\partial z^{2}}+\xi\frac{\partial V}{\partial x}\left(\frac{\partial^{2}U}{\partial z^{2}}+\frac{\partial^{2}W}{\partial z\partial x}\right)+2\xi\frac{\partial^{2}W}{\partial z^{2}}\\
&\frac{\partial V}{\partial z}+2\xi\frac{\partial W}{\partial z}\frac{\partial^{2}V}{\partial z^{2}}\qquad \text{…. (2-12)}
\end{aligned}
$$

$$
\begin{aligned}
\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+W\frac{\partial W}{\partial z}\right)&=-\frac{\partial P}{\partial z}+\eta\left(\frac{\partial^{2}U}{\partial x\partial z}+\frac{\partial^{2}W}{\partial x^{2}}\right)+\\
&2\xi\frac{\partial^{2}U}{\partial x^{2}}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)+2\xi\frac{\partial^{2}W}{\partial x\partial z}\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)+2\xi\frac{\partial W}{\partial z}\\
&\left(\frac{\partial^{2}U}{\partial x\partial z}+\frac{\partial^{2}W}{\partial x^{2}}\right)+2\eta\frac{\partial^{2}W}{\partial z^{2}}+2\xi\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)\left(\frac{\partial^{2}U}{\partial z^{2}}+\frac{\partial^{2}W}{\partial x\partial z}\right)\\
&+2\xi\frac{\partial V}{\partial z}\frac{\partial^{2}V}{\partial z^{2}}+8\xi\frac{\partial W}{\partial z}\frac{\partial^{2}W}{\partial z^{2}}\qquad \text{…. (2-13)}
\end{aligned}
$$

The boundary conditions are *U=V=W=0 on the boundary*. …. (2-14)

By using the stream function, equation (1-7), for the velocity components U, W and eliminating the pressure from equations (2-11) and (2-13) we obtain

$$\rho\left(-\frac{\partial}{\partial t}\nabla^2\psi+\frac{\partial\psi}{\partial z}\frac{\partial^3\psi}{\partial^2 z\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial^3 z}-\frac{2}{R}V\frac{\partial V}{\partial z}+\right.$$

$$\left.\frac{\partial\psi}{\partial z}\frac{\partial^3\psi}{\partial^3 x}-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial z\partial^2 x}\right)=2\eta\frac{\partial^4\psi}{\partial z^2\partial x^2}-\eta\frac{\partial^4\psi}{\partial z^4}-$$

$$2\frac{\xi}{R}\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial z\partial x}-2\frac{\xi}{R}\frac{\partial V}{\partial z}\frac{\partial^2 V}{\partial z^2}-\eta\frac{\partial^4\psi}{\partial x^4} \qquad \text{…. (2-15)}$$

And equation (2-12) become

$$\rho\left(\frac{\partial V}{\partial t}-\frac{\partial\psi}{\partial z}\frac{\partial V}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial V}{\partial z}\right)=-\frac{1}{R}\frac{\partial P}{\partial\theta}+\eta\nabla^2 V-2\xi\frac{\partial^3\psi}{\partial x^2\partial z}\frac{\partial V}{\partial x}-$$

$$2\xi\frac{\partial^2\psi}{\partial x\partial z}\frac{\partial^2 V}{\partial x^2}-\xi\frac{\partial^3\psi}{\partial x\partial z^2}\frac{\partial V}{\partial x}+\xi\frac{\partial^3\psi}{\partial x^3}\frac{\partial V}{\partial z}-2\xi\frac{\partial^2\psi}{\partial z^2}\frac{\partial^2 V}{\partial x\partial z}+$$

$$2\xi\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2 V}{\partial x\partial z}-\xi\frac{\partial^3\psi}{\partial z^3}\frac{\partial V}{\partial x}+\ 2\frac{\xi}{R}\frac{\partial^2\psi}{\partial z^2}\frac{\partial V}{\partial z}-2\frac{\xi}{R}\frac{\partial\psi}{\partial z}\frac{\partial^2 V}{\partial z^2}+$$

$$2\xi\frac{\partial^2\psi}{\partial x\partial z}\frac{\partial^2 V}{\partial z^2}+\xi\frac{\partial^3\psi}{\partial z\partial x^2}\frac{\partial V}{\partial x}+2\xi\frac{\partial^3\psi}{\partial z^2\partial x}\frac{\partial V}{\partial z} \qquad \text{…. (2-16)}$$

The last equations, (2-15) and (2-16), can be simplified to

$$\upsilon\nabla^4\psi = \frac{\partial}{\partial t}\nabla^2\psi + \frac{2}{R}V\frac{\partial V}{\partial z} + \left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2\psi$$
$$-2\frac{\xi}{\rho R}\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x\partial z} - 2\frac{\xi}{\rho R}\frac{\partial V}{\partial z}\frac{\partial^2 V}{\partial z^2} \qquad \text{.... (2-17)}$$

$$\upsilon\nabla^2 V - \frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{P}{\rho}\right) = \frac{\partial V}{\partial t} + \left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\right)V +$$
$$\frac{\xi}{\rho}\left(\frac{\partial V}{\partial x}\frac{\partial}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2\psi + 2\frac{\xi}{\rho}\left(\frac{\partial^2 V}{\partial x\partial z}\right)\left(\frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\psi}{\partial x^2}\right) +$$
$$2\frac{\xi}{\rho}\left(\frac{\partial^2\psi}{\partial x\partial z}\right)\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial z^2}\right) \qquad \text{.... (2-18)}$$

$$\left.\begin{array}{l} \psi = \dfrac{\partial\psi}{\partial x} = \dfrac{\partial\psi}{\partial z} = 0 \text{ , on the boundary} \\ v = 0, \text{ on the boundary} \end{array}\right\} \qquad \text{.... (2-19)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial z^2} + \frac{\partial^4}{\partial z^4}$$

We impose a sinusoidal pressure gradient in time with zero mean on the flow in the form of

$$-\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{P}{\rho}\right) = J\ V_o\alpha Cos(\alpha t) \qquad \text{.... (2-20)}$$

for convenient computation we will choose $J = 2.31$.
where $JVo\alpha$ is the amplitude of the applied pressure gradient and α is the angular frequency.

2.4 Non-Dimensional Form of Motion Equation for the Case1

It is possible to write the motion equations (2-17) - (2-19) in non-dimensional form through using the following new quantities

$$x_1 = \frac{x}{d},\ z_1 = \frac{z}{d},\ \tau = t\alpha\ ,\ f = \frac{\psi}{v},\ \ v = \frac{V}{V_0} \qquad \text{.... (2-21)}$$

Equations (2-17) - (2-19), then become

$$\nabla^4 f = \kappa^2 \frac{\partial}{\partial t}\nabla^2 f + Lv\frac{\partial v}{\partial z_1} + \left(\frac{\partial f}{\partial x_1}\frac{\partial}{\partial z_1} - \frac{\partial f}{\partial z_1}\frac{\partial}{\partial x_1}\right)\nabla^2 f -$$

$$\beta L\left(\frac{\partial v}{\partial x_1}\frac{\partial^2 v}{\partial x_1 \partial z_1} + \frac{\partial v}{\partial z_1}\frac{\partial^2 v}{\partial {z_1}^2}\right) \qquad \text{.... (2-22)}$$

$$\nabla^2 v = -2.31\kappa^2 Cos(\tau) + \kappa^2 \frac{\partial v}{\partial \tau} + \left(\frac{\partial f}{\partial x_1}\frac{\partial}{\partial z_1} - \frac{\partial f}{\partial z_1}\frac{\partial}{\partial x_1}\right)v +$$

$$\beta\left(\frac{\partial v}{\partial x_1}\frac{\partial}{\partial z_1} - \frac{\partial v}{\partial z_1}\frac{\partial}{\partial x_1}\right)\nabla^2 f + 2\beta \frac{\partial^2 V}{\partial x_1 \partial z_1}\left(\frac{\partial^2 f}{\partial {z_1}^2} - \frac{\partial^2 f}{\partial {x_1}^2}\right) +$$

$$2\beta \frac{\partial^2 f}{\partial x_1 \partial z_1}\left(\frac{\partial^2 v}{\partial {x_1}^2} - \frac{\partial^2 v}{\partial {z_1}^2}\right) \qquad \text{.... (2-23)}$$

$$\left.\begin{aligned} & f = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial z_1} = 0, \ \ \textit{on the boundary} \\ & v = 0, \ \ \textit{on the boundary} \end{aligned}\right\} \qquad \text{.... (2-24)}$$

These equations can be seen to be controlled by three parameters, a non-dimensional frequency parameter, $k = d\left(\frac{\alpha}{\upsilon}\right)^2$, the non-Newtonian parameter $\beta = \frac{\xi}{\rho d^2}$ and Dean number $L = \frac{2V_0^2 d^3}{R\upsilon^2}$.

In what follows we shall omit the index of coordinate system, it is understood that all variables are in non-dimensional form. To solve the above system, (2-22)-(2-24), we will use successive approximation method, which is equivalent to the perturbation solutions of f and v in ascending powers of L. So the solution of the above system can be developed by using

$$f(x,z,t) = Lf_1(x,z,t) + L^2 f_2(x,z,t) + \cdots$$
$$v(x,z,t) = v_o(x,z,t) + Lv_1(x,z,t) + L^2 v_2(x,z,t) + \cdots \qquad ...(2\text{-}25)$$

Where $f_0(x,z,t) = 0$ in a straight pipe. We will limit ourselves to find the solution up to the first order in L, similar procedures can be used for higher order solutions, and the first order solution provide goods accuracy for the purpose. If we substitute (2-25) in (2-22) - (2-24), and equate coefficients of equal powers in L; we obtain a series of relations from which v_o, f_1, v_1,...can be successively found .The equations are

$$\nabla^2 v_o = \kappa^2 \frac{\partial v_o}{\partial \tau} - 2.31\kappa^2 Cos(\tau) \qquad .... (2\text{-}26)$$

$$\nabla^4 f_1 = \kappa^2 \frac{\partial}{\partial \tau}\nabla^2 f_1 + v_o \frac{\partial v_o}{\partial z} - \beta\left(\frac{\partial v_o}{\partial x}\frac{\partial^2 v_o}{\partial x \partial z} + \frac{\partial v_o}{\partial z}\frac{\partial^2 v_o}{\partial z^2}\right) \qquad ...(2\text{-}27)$$

$$\nabla^2 v_1 = \kappa \frac{\partial v_1}{\partial \tau} + \left(\frac{\partial f_1}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial}{\partial x}\right)v_o + \beta\left(\frac{\partial v_o}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_o}{\partial z}\frac{\partial}{\partial x}\right)$$
$$\nabla^2 f_1 + 2\beta \frac{\partial^2 v_o}{\partial x \partial z}\left(\frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_1}{\partial x^2}\right) + 2\beta \frac{\partial^2 f_1}{\partial x \partial z}\left(\frac{\partial^2 v_o}{\partial x^2} - \frac{\partial^2 v_o}{\partial z^2}\right) \qquad ...(2\text{-}28)$$

The boundary conditions associated with the above equations, (2-26) – (2-28) are: -

$$\left.\begin{aligned} f_1 = \frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial z} = 0 \quad & \textit{on the boundary} \\ v_n = 0\ , n = o,1,\cdots \quad & \textit{on the boundary} \end{aligned}\right\} \qquad .... (2\text{-}29)$$

2.5 Non-Dimensional Form of Motion Equation for the Case2

By similar procedure, with exception that h is the characteristic length instead of d, the non-dimensional parameters are defined as

$$x_1 = \frac{x}{h},\ z_1 = \frac{z}{h},\ \tau = t\alpha,\ f = \frac{\psi}{\nu},\ \ v = \frac{V}{V_0} \qquad \text{…. (2-30)}$$

Equations (2-17) - (2-19) become

$$\nabla^2 v_o = \kappa^2 \frac{\partial v_o}{\partial \tau} - 2.31\kappa^2 Cos(\tau) \qquad \text{…. (2-31)}$$

$$\nabla^4 f_1 = \kappa^2 \frac{\partial}{\partial \tau}\nabla^2 f_1 + v_o \frac{\partial v_o}{\partial z} - \beta\left(\frac{\partial v_o}{\partial x}\frac{\partial^2 v_o}{\partial x \partial z} + \frac{\partial v_o}{\partial z}\frac{\partial^2 v_o}{\partial z^2}\right) \qquad \text{...(2-32)}$$

$$\nabla^2 v_1 = \kappa \frac{\partial v_1}{\partial \tau} + \left(\frac{\partial f_1}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial}{\partial x}\right)v_o + \beta\left(\frac{\partial v_o}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_o}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f_1$$
$$+ 2\beta \frac{\partial^2 v_o}{\partial x \partial z}\left(\frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_1}{\partial x^2}\right) + 2\beta \frac{\partial^2 f_1}{\partial x \partial z}\left(\frac{\partial^2 v_o}{\partial x^2} - \frac{\partial^2 v_o}{\partial z^2}\right) \qquad \text{... (2-33)}$$

where $k = h\left(\frac{\alpha}{\upsilon}\right)^2$, $\beta = \frac{\xi}{\rho h^2}$ and $L = \frac{2V_0^2 h^3}{R\upsilon^2}$, and the boundary conditions associated with this system, (2-31)-(2-33), are

$$\left.\begin{aligned} & f_1 = \frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial z} = 0 \quad && \textit{on the boundary} \\ & v_n = 0\ , n = o,1,\cdots && \textit{on the boundary} \end{aligned}\right\} \qquad \text{…. (2-34)}$$

Chapter Three

A Variational Method for Solving the Problem

Introduction

In this chapter, an approximate solution to the problem is obtained through a variational method, Galerkin's method, [4], [12], [25], for both cases.

Actually, the variational methods including Galerkin give an analytic approximate solution for partial differential equations which describe a fluid mechanics problem. Since it is difficult to find an exact solution we resort to consider approximate solution for these equations.

This chapter, also include the solutions of steady state for the two cases under consideration.

3.1 Galerkin's Method

In 1915, B.G. Galerkin presented a new variational method to solve boundary value problems which was of a wide interest to researchers in the field of Applied Mathematics and Engineering Applications.

The method is summarized in finding the solution of the equation $L(u) = f$, $u \in L_2(G)$, where L is a differential operator in two variables and f is a given continuous function in two variables defined on a region G. We shall seek an approximate solution of the problem in the form

$$u_n(x, y) = \sum_{i=1}^{n} c_i \Phi_i(x, y) \qquad \text{.... (3-1)}$$

Where $\{\Phi_i(x,y), i=1,2,3..n\}$ is a system of functions (which is usually, called a base of coordinates) chosen before hand and is satisfying a certain conditions

a) It should be linearly independent in $L_2(G)$.
b) It should be complete in this space.

And the coefficients c_i are to be determined. Our aim is to find the c_i values such that $u_n(x,y)$ is close to the exact solution in the sense that $Lu_n - f$ is orthogonal to Φ_i, $i=1,2,3\ldots,n$. i.e.

$$\left.\begin{aligned}&\iint_G [Lu_n(x,y) - f(x,y)]\Phi_i(x,y)dxdy = 0\\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1,\ldots,n\\ &\iint_G \left[L\left(\sum_{j=0}^{n} c_j\Phi_j(x,y)\right) - f(x,y)\right]\Phi_i(x,y)dxdy = 0\end{aligned}\right\} \quad \ldots\ldots(3\text{-}2)$$

This is an algebraic system of equations for the unknowns c_i, $i=1,2,3,\ldots,n$ when we solve the above system by one of the direct numerical methods like "Gauss elimination, Gauss Jordan" or iterative numerical methods like "Gauss-sidel method or successive over-relaxation method" we get c_i and substitute in (3.1) to get the n^{th} approximate solution $u_n(x,y)$.[25]

3.2 solution of Case1

Galerkin's method is employed to solve the equations (2-26)-(2-28) subjected to the associated boundary conditions (2-29).

3.2.1 Solution for v_0:

If we substitute for v_0 in equation (2-26) by the expression

$$v_o = \kappa^2 v_{01} + \kappa^4 v_{02} + \kappa^6 v_{03} + \kappa^8 v_{04} + O(\kappa^{10}) \qquad \ldots.\ (3\text{-}3)$$

and equate the coefficient of equal powers in k for equation (2-26), then the following set of equations are obtained

$$\nabla^2 v_{01} = -2.31 Cos(\tau) \qquad \text{.... (3-4)}$$

$$\nabla^2 v_{02} = \frac{\partial v_{01}}{\partial \tau} \qquad \text{.... (3-5)}$$

$$\nabla^2 v_{03} = \frac{\partial v_{02}}{\partial \tau} \qquad \text{.... (3-6)}$$

$$\nabla^2 v_{04} = \frac{\partial v_{03}}{\partial \tau} \qquad \text{.... (3-7)}$$

with $v_{0i} = 0$, *i=1,2,3,4 on the boundary*. ... (3-8)

Solution of (3-4) can be developed by assuming that

$$v_{01} = v_{011}(x,z) Cos(\tau) \qquad \text{.... (3-9)}$$

If substitute equation (3-9) in (3-4) we get

$$\nabla^2 v_{011} = -2.31 \qquad \text{.... (3-10)}$$

So the employed Galerkin's method is equivalent to the assuming of solution in the form

$$v_{011} = a_0 \left(1 - x^2\right)\left(\frac{4}{9} - z^2\right) \qquad \text{.... (3-11)}$$

where a_0 is a constant to be determined. It is found that the solution of (3-11) is

$$v_{011} = \left(1 - x^2\right)\left(\frac{4}{9} - z^2\right) \qquad \text{.... (3-12)}$$

Thus the complete zeroth order solution is

$$v_{01} = \left(1 - x^2\right)\left(\frac{4}{9} - z^2\right) Cos(\tau) \qquad \text{.... (3-13)}$$

If we substitute equation v_{01} in equation (3-5) and using the procedure of Galerkin's method, the solution of v_{02} is found to be of the form

$$v_{02} = \left(1 - x^2\right)\left(\frac{4}{9} - z^2\right)\left(a_1 + a_2 x^2 + a_3 z^2 + a_4 x^2 z^2\right) Sin(\tau) \quad ...(3\text{-}14)$$

where a_1, a_2, a_3 and a_4 are constant.

Similarly, solution for v_{03} and v_{04} can be found. Finally zero order solution for v_0 thus obtained.

The substituting of these solutions into equation (3-3) give the solution for v_0 which is

$$\begin{aligned} v_o = {} & \kappa^2 (1 - x^2)(\frac{4}{9} - z^2)[Cos(\tau) + \kappa^2 (a_1 + a_2 x^2 + a_3 z^2 + \\ & a_4 x^2 z^2) Sin(\tau) + \kappa^4 (b_1 + b_2 x^2 + b_3 z^2 + b_4 x^2 z^2 + b_5 x^4 + \\ & b_6 x^4 z^2 + b_7 z^4 + b_8 x^2 z^4 + b_9 x^4 z^4) Cos(\tau) + \kappa^6 (c_1 + c_2 x^2 + \\ & c_3 z^2 + c_4 x^2 z^2 + c_5 x^4 + c_6 x^4 z^2 + c_7 z^4 + c_8 x^2 z^4 + c_9 x^4 z^4 + \\ & c_{10} x^6 + c_{11} x^6 z^2 + c_{12} x^6 z + c_{13} z^6 + c_{14} x^2 z^6 + c_{15} x^4 z^6 + \\ & c_{16} x^6 z^6) Sin(\tau)] + O(\kappa^{10}) \qquad \ldots(3\text{-}15) \end{aligned}$$

<u>3.2.2 Solution for f_1:</u>

The equation (2-27) contains the function v_0, which is now known through the solution (3-15). If we substitute of v_0 into (2-27), then that equation will contain only one unknown function which is f_1, the solution for f_1 is obtained as a perturbation in terms of the parameter κ as follows: -

$$f_1 = \kappa^4 f_{11} + \kappa^6 f_{12} + O(\kappa^8) \qquad \ldots. (3\text{-}16)$$

The recursive equations for f_{1i}, i=1, 2 are obtained on equating the coefficients of equal powers in κ these equations are

$$\nabla^4 f_{11} = v_{01} \frac{\partial v_{01}}{\partial z} - \beta \left(\frac{\partial v_{01}}{\partial x} \frac{\partial^2 v_{01}}{\partial x \partial z} + \frac{\partial v_{01}}{\partial z} \frac{\partial^2 v_{01}}{\partial z^2} \right) \qquad \ldots. (3\text{-}17)$$

$$\nabla^4 f_{12} = \frac{\partial}{\partial \tau}\nabla^2 f_{11} + v_{01}\frac{\partial v_{02}}{\partial z} + v_{02}\frac{\partial v_{01}}{\partial z} - \beta\left(\frac{\partial v_{01}}{\partial x}\frac{\partial^2 v_{02}}{\partial x \partial z} + \frac{\partial v_{02}}{\partial x}\frac{\partial^2 v_{01}}{\partial x \partial z} + \frac{\partial v_{01}}{\partial z}\frac{\partial^2 v_{02}}{\partial z^2} + \frac{\partial v_{02}}{\partial z}\frac{\partial^2 v_{01}}{\partial z^2}\right) \qquad \text{…. (3-18)}$$

and the boundary conditions are:

$$\left.\begin{aligned} & f_{1i} = \frac{\partial f_{1i}}{\partial x} = \frac{\partial f_{1i}}{\partial z} = 0,\ i = 1,2 \textit{ on the bondary} \\ & v_{0i} = 0,\ i = 1,2 \textit{ on the bondary} \end{aligned}\right\} \qquad \text{…. (3-19)}$$

Again, we proceed to eliminate the time variable and generate a solution as an expansion in non-dimensional parameter β.The solution for f_1 is found to be of the form

$$f_1 = \kappa^4\left(f_{111} + \beta f_{112}\right)Cos^2(\tau) + \kappa^6\left(f_{121} + \beta f_{122}\right)Cos(\tau)\, Sin(\tau) + O(\kappa^8) \qquad \text{…. (3-20)}$$

<u>3.2.3 Solution for v_1:</u>

Similarly, we assume that

$$v_1 = \kappa^6 v_{11} + \kappa^8 v_{12} + O(\kappa^{10}) \qquad \text{.... (3-21)}$$

The solution (3-21) is substituted into (2-28) and we make use of the solutions (3-15) and (3-20), the recursive equations are

$$\nabla^2 v_{11} = \left(\frac{\partial f_{11}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_{11}}{\partial z}\frac{\partial}{\partial x}\right)v_{01} + \beta\left(\frac{\partial v_{01}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_{01}}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f_{11} + 2\beta\frac{\partial^2 v_{01}}{\partial x \partial z}\left(\frac{\partial^2 f_{11}}{\partial z^2} - \frac{\partial^2 f_{11}}{\partial x^2}\right) + 2\beta\frac{\partial^2 f_{11}}{\partial x \partial z}\left(\frac{\partial^2 v_{01}}{\partial x^2} - \frac{\partial^2 v_{01}}{\partial z^2}\right) \quad \ldots(3\text{-}22)$$

$$\nabla^2 v_{12} = \frac{\partial v_{11}}{\partial \tau} + \left(\frac{\partial f_{11}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_{11}}{\partial z}\frac{\partial}{\partial x}\right)v_{02} + \left(\frac{\partial f_{12}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_{12}}{\partial z}\frac{\partial}{\partial x}\right)v_{01} + \beta\left(\frac{\partial v_{01}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_{01}}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f_{12} + \beta\left(\frac{\partial v_{02}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_{02}}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f_{11} + 2\beta\frac{\partial^2 v_{01}}{\partial x \partial z}\left(\frac{\partial^2 f_{12}}{\partial z^2} - \frac{\partial^2 f_{12}}{\partial x^2}\right) + 2\beta\frac{\partial^2 v_{02}}{\partial x \partial z}\left(\frac{\partial^2 f_{11}}{\partial z^2} - \frac{\partial^2 f_{11}}{\partial x^2}\right) + 2\beta\frac{\partial^2 f_{11}}{\partial x \partial z}\left(\frac{\partial^2 v_{02}}{\partial x^2} - \frac{\partial^2 v_{02}}{\partial z^2}\right) + 2\beta\frac{\partial^2 f_{12}}{\partial x \partial z}\left(\frac{\partial^2 v_{01}}{\partial x^2} - \frac{\partial^2 v_{01}}{\partial z^2}\right) \ldots(3\text{-}23)$$

the boundary condition are

$$\left.\begin{aligned} f_{1i} &= \frac{\partial f_{1i}}{\partial x} = \frac{\partial f_{1i}}{\partial z} = 0,\ i = 1,2 \ \textit{on the bondary} \\ v_{1i} &= 0,\ i = 1,2 \ \textit{on the bondary} \end{aligned}\right\} \quad \ldots.\ (3\text{-}24)$$

By similar procedure the solution for v_1 is found to be of the form

$$v_1 = \kappa^6\left[\left(v_{111} + \beta v_{112} + \beta^2 v_{113}\right)Cos^3(\tau) + \kappa^2\left(v_{121} + \beta v_{122} + \beta^2 v_{123}\right)Cos^2(\tau)Sin(\tau) + O(\kappa^4)\right] \ldots(3\text{-}25)$$

Finally, substitute the solutions v_0, f_1 and v_1 into (2-25), the stream function and the axial velocity can be written in a convent form

$$f(x,z,\tau) = L f_1(x,z,\tau)$$
$$f(x,z,\tau) = L[\kappa^4(f_{111} + \beta f_{112})Cos^2(\tau) + \kappa^6(f_{121} + \beta f_{122}) Cos(\tau)Sin(\tau)] \quad \ldots.\ (3\text{-}26)$$

$$v = v_0 + Lv_1$$

$$v = \kappa^2 v_{01} + k^4 v_{02} + \kappa^6 v_{03} + k^8 v_{04} + L\kappa^6 (v_{111} + \beta v_{112} + \beta^2 v_{113}) Cos^3(\tau) +$$
$$L\kappa^8 (v_{121} + \beta v_{122} + \beta^2 v_{123})\} Cos(\tau)^2 Sin(\tau) \qquad .... (3\text{-}27)$$

where all the above $f's$ and $v's$ are polynomials in x and z.

If *f* and *v* are independent of t and $k = 1$ the system (2-22) - (2-24) will be reduced to corresponding system in case of steady state, which is

$$\nabla^4 f = Lv\frac{\partial v}{\partial z} + \left(\frac{\partial f}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f - \beta L\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 v}{\partial x \partial z}\right) -$$
$$\beta L\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial^2 v}{\partial z^2}\right) \qquad ....(3\text{-}28)$$

$$\nabla^2 v = -2.31 + \left(\frac{\partial f}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial x}\right)v + \beta\left(\frac{\partial v}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f +$$
$$2\beta\left(\frac{\partial^2 v}{\partial x \partial z}\right)\left(\frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial x^2}\right) + 2\beta\left(\frac{\partial^2 f}{\partial x \partial z}\right)\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial z^2}\right) \qquad ....(3\text{-}29)$$

and the associated boundary conditions are:

$$\left.\begin{array}{l} f = \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial z} = 0, \ on the boundary \\ v = 0, \ on the boundary \end{array}\right\} \qquad .... (3\text{-}30)$$

In substituting (2-25) in system (3-28) - (3-30), and equate coefficients of equal power in *L*, we obtain

$$\nabla^2 v_o = -2.31 \qquad \text{.... (3-31)}$$

$$\nabla^4 f_1 = v_o \frac{\partial v_o}{\partial z} - \beta \frac{\partial v_o}{\partial x}\frac{\partial^2 v_o}{\partial x \partial z} - \beta \frac{\partial v_o}{\partial z}\frac{\partial^2 v_o}{\partial z^2} \qquad \text{.... (3-32)}$$

$$\nabla^2 v_1 = \left(\frac{\partial f_1}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial}{\partial x}\right)v_o + \beta\left(\frac{\partial v_0}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_o}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f_1 +$$
$$2\beta\left(\frac{\partial^2 v_o}{\partial x \partial z}\right)\left(\frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_1}{\partial x^2}\right) + 2\beta\left(\frac{\partial^2 f_1}{\partial x \partial z}\right)\left(\frac{\partial^2 v_o}{\partial x^2} - \frac{\partial^2 v_o}{\partial z^2}\right) \qquad \text{.... (3-33)}$$

The boundary conditions associated with system (3-31) – (3-33), are:

$$\left.\begin{array}{ll} f_{1i} = \dfrac{\partial f_{1i}}{\partial x} = \dfrac{\partial f_{1i}}{\partial z} = 0,\ i = 1,2 & on\, the\, boundary \\ v_n = 0,\ n = 0,1 & on\, the\, boundary \end{array}\right\} \qquad \text{.... (3-34)}$$

The solution of system (3-31) – (3-33) subjected to the boundary condition (3-34) is

$$f(x,z,\tau) = L f_1(x,z,\tau)$$

$$f(x,z,\tau) = L((1-x^2)(\frac{4}{9}-z^2))^2\big[(e_1 z + e_2 x^2 z + e_3 z^3 + e_4 x^2 z^3 +$$
$$e_5 x^4 z + e_6 x^4 z^3) + \beta(g_1 x^2 z + g_2 x^2 z^3 + g_3 z + g_4 x^4 z)\big] \qquad \text{.... (3-35)}$$

where $e_1,\ldots,e_6$, $g_1,\ldots,g_4$ *are real const*.

$$v = v_0 + L v_1$$

$$v = (1-x^2)(\frac{4}{9}-z^2)\big[1 + L(v_{11} + \beta v_{12} + \beta^2 v_{13})\big] \qquad \text{.... (3-36)}$$

In addition to that if we set $\beta = 0$ in (3-35) and (3-36) we obtained the solution in case of Newtonian fluid. [10]

3.3 Solution of case2

By similar procedure the solution of motion equations for case2 is found and Galerkin's method is employed.

3.3.1 Solution for v_o:

We assume the solution of v_0 is

$$v_o = \kappa^2 v_{01} + \kappa^4 v_{02} + \kappa^6 v_{03} + \kappa^8 v_{04} + O(\kappa^{10}) \qquad \text{…. (3-37)}$$

where

$$v_{01} = \left(\frac{4}{9} - x^2\right)\left(1 - z^2\right)Cos(\tau) \qquad \text{…. (3-38)}$$

$$v_{02} = \left(\frac{4}{9} - x^2\right)\left(1 - z^2\right)\left(i_1 + i_2 x^2 + i_3 z^2 + i_4 x^2 z^2\right)Sin(\tau) \qquad \text{...(3-39)}$$

The solution for v_{03} and v_{04} are obtained by the same way. Thus the solution for v_0 is

$$\begin{aligned} v_o = {} & \kappa^2 (\frac{4}{9} - x^2)(1 - z^2)[Cos(\tau) + \kappa^2 (i_1 + i_2 x^2 + i_3 z^2 + \\ & i_4 x^2 z^2)Sin(\tau) + \kappa^4 (j_1 + j_2 x^2 + j_3 z^2 + j_4 x^2 z^2 + j_5 x^4 + \\ & j_6 x^4 z^2 + j_7 z^4 + j_8 x^2 z^4 + j_9 x^4 z^4)Cos(\tau) + \kappa^6 (m_1 + m_2 x^2 + \\ & m_3 z^2 + m_4 x^2 z^2 + m_5 x^4 + m_6 x^4 z^2 + m_7 z^4 + m_8 x^2 z^4 + m_9 x^4 z^4 + \\ & m_{10} x^6 + m_{11} x^6 z^2 + m_{12} x^6 z + m_{13} z^6 + m_{14} x^2 z^6 + m_{15} x^4 z^6 + \\ & m_{16} x^6 z^6)Sin(\tau)] + O(\kappa^{10}) \end{aligned} \qquad \text{…. (3-40)}$$

3.3.2 Solution for f_1:

The solution for f_1 is found to be

$$f_1 = \kappa^4\left(f_{111} + \beta f_{112}\right)Cos^2(\tau) + \kappa^6\left(f_{121} + \beta f_{122}\right)$$
$$Cos(\tau)Sin(\tau) + O(\kappa^8) \qquad \text{.... (3-41)}$$

3.3.3 Solution for v_1:

Similarly, the solution for v_1 is

$$v_1 = \kappa^6\left(v_{111} + \beta v_{112} + \beta^2 v_{113}\right)Cos^3(\tau) + \kappa^8\left(v_{121} + \beta v_{122} + \beta^2 v_{123}\right)$$
$$Cos^2(\tau)Sin(\tau) + O(\kappa^{10}) \qquad \text{.... (3-42)}$$

Finally, in substituting the solution v_0, f_1 and v_1 into (2-25), the stream function and the axial velocity can be written in a convent form

$$f(x,z,\tau) = L f_1(x,z,\tau)$$

$$f(x,z,\tau) = L[\kappa^4(f_{111} + \beta\ f_{112})Cos^2(\tau) + \kappa^6(f_{121} + \beta\ f_{122})$$
$$Cos(\tau)Sin(\tau)] \qquad \text{.... (3-43)}$$

$$v = v_0 + L v_1$$
$$v = \kappa^2 v_{01} + k^4 v_{02} + \kappa^6 v_{03} + k^8 v_{04} + L\kappa^6(v_{111} + \beta v_{112} + \beta^2 v_{113})Cos^3(\tau) +$$
$$L\kappa^8(v_{121} + \beta v_{122} + \beta^2 v_{123})Cos(\tau)^2 Sin(\tau) \qquad \text{.... (3-44)}$$

where all the above $f's$ and $v's$ are polynomials in x and z.

If f and v are independent of t and $k = 1$, then the liner motion equations for the case of steady state, are

$$\nabla^2 v_o = -2.31 \qquad \text{.... (3-45)}$$

$$\nabla^4 f_1 = v_o \frac{\partial v_o}{\partial z} - \beta \frac{\partial v_o}{\partial x}\frac{\partial^2 v_o}{\partial x \partial z} - \beta \frac{\partial v_o}{\partial z}\frac{\partial^2 v_o}{\partial z^2} \qquad \text{.... (3-46)}$$

$$\nabla^2 v_1 = \left(\frac{\partial f_1}{\partial x}\frac{\partial}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial}{\partial x}\right)v_o + \beta\left(\frac{\partial v_0}{\partial x}\frac{\partial}{\partial z} - \frac{\partial v_o}{\partial z}\frac{\partial}{\partial x}\right)\nabla^2 f_1 +$$
$$2\beta\left(\frac{\partial^2 v_o}{\partial x \partial z}\right)\left(\frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_1}{\partial x^2}\right) + 2\beta\left(\frac{\partial^2 f_1}{\partial x \partial z}\right)\left(\frac{\partial^2 v_o}{\partial x^2} - \frac{\partial^2 v_o}{\partial z^2}\right) \qquad \text{.... (3-47)}$$

The boundary conditions associated with system (3-45) - (3-47), are:

$$\left.\begin{array}{ll} f_{1i} = \dfrac{\partial f_{1i}}{\partial x} = \dfrac{\partial f_{1i}}{\partial z} = 0,\ i = 1,2 \ \ on the boundary & \\ v_n = 0,\ \ n = 1,2 \qquad\qquad\qquad\qquad on the boundary & \end{array}\right\} \qquad ...(3\text{-}48)$$

And the solution of system (3-45) – (3-47) subjected to the boundary condition (3-48), is

$$f(x,z,\tau) = L f_1(x,z,\tau)$$

$$f(x,z,\tau) = L((\frac{4}{9} - x^2)(1-z^2))^2\big[(q_1 z + q_2 x^2 z + q_3 z^3 + q_4 x^2 z^3 + q_5 x^4 z + q_6 x^4 z^3) + \beta(r_1 x^2 z + r_2 x^2 z^3 + r_3 z + r_4 x^4 z)\big] \qquad .... (3\text{-}49)$$

where $q_1,\ldots,q_6$, $r_1,\ldots,r_4$ *are real const*.

$$v = v_0 + L v_1$$

$$v = (\frac{4}{9} - x^2)(1-z^2)\big[1 + L(v_{11} + \beta v_{12} + \beta^2 v_{13})\big] \qquad .... (3\text{-}50)$$

Also, if we set $\beta = 0$ in (3-49) and (3-50), we obtained the solution in case of Newtonian fluid. [10]

CHAPTER FOUR

Results and Discussion

Introduction

In this chapter, the analysis of the solutions, for both cases, is considered. The effect of parameters that control the motion equations on various important flow characteristic, (i.e. the secondary flow and the axial velocity) is studied for different values of these parameters.

We explain the effect of these parameters through drawing the projection of streamline in the central plane and in the cross-section of the pipe. A comparison between the values of stream function and the value of the axial velocity, for both cases, is given.

Also, in our analysis we consider the case of flow of Newtonian fluid in curved pipes.

4.1 Secondary Flow

The secondary flow occurs in curved ducts or curved pipes. Physically the parameter L (Dean number) can be considered as the ratio of the centrifugal force induced by circular motion of the fluid to viscous force when a fluid flows through a curved pipe. Pressure gradient directed towards the center of curvature, is setup across the pipe to balance the centrifugal force arising from curvature. The fluid near the wall of the pipe is moving more slowly than the fluid some way from the wall owing to viscosity and therefore require small pressure gradient to balance the local centrifugal force. As a result of these different pressure gradients, the faster-flowing fluid moves outwards, whilst the slower-flowing fluid moves inward.

This flow is known as the secondary flow and it is superposed on the main stream region towards the outer wall and creating a much thicker layer of slowly moving fluid at the inner wall, however, owing the enhanced mixing and momentum transfer due to the secondary flow, the total frictional loss of energy near the wall increases and the fluid experiences more resistance in posing through the pipe.

4.2 Streamline Projection for Case1

The differential equations of the streamline is, [29]

$$\frac{dX}{U} = \frac{(R+x)d\theta}{V} = \frac{dZ}{W} \qquad \text{…. (4-1)}$$

The velocity components, *(U, V, W)* are to be obtained from equations (3-35) and (3-36).

Up to sufficient accuracy equation (4-1) may be written as

$$\frac{dX}{U} = \frac{Rd^4 d\theta}{V_o\left(1-x^2\right)\left(\frac{h^2}{d^2}-z^2\right)} = \frac{dZ}{W} \qquad \text{…. (4-2)}$$

It is clear that all the variables are in the dimensional form.

4.2.1 Streamline Projection in the Central Pane:

The motion of the liquid in the central plane of the pipe is of special simplicity .At any point on OC we have z = 0 and $\partial\psi / \partial x = 0$, -1 ≤ x ≤ 1 which mean that w vanishes; (i.e. the liquid particles located in the central plane do not possess the w component of velocity which is responsible of moving them out of this (x = 0) plane). As a result the direction of the velocity at such point in the liquid lies in the central plane. Thus the motion in the upper half of the pipe is quite distinct from that in the lower half and it is clear that the central plane is the plane of symmetry for the motion.

The differential equation of the streamline in the central plane is

$$\frac{dx}{U} = \frac{R d^2 d\theta}{V_o \frac{h^2}{d^2}\left(d^2 - x^2\right)} \qquad \text{…. (4-3)}$$

From the dimensional analysis we have

$$U = \frac{v\, u}{d} \qquad \text{…. (4-4)}$$

Then by using equations (4-4) and (2-25) we obtain

$$U = \frac{v\, L}{d}\frac{\partial f_1}{\partial z} \qquad at \;\; z = 0 \qquad \text{…. (4-5)}$$

where $L = 2 R_e^2 \left(\frac{d}{R}\right)$

Substituting equation (4-5) into equation (4-3) we obtain

$$\frac{dx_1}{d\theta} = \frac{-2 R_e}{.\frac{h^2}{d^2}\left(1 - x_1^2\right)} . \frac{\partial f_1}{\partial z}\Big|_{z=0} \qquad \text{…. (4-6)}$$

where $R_e = V_o\, d / \upsilon$, is Reynolds number which determine the nature of flow.

Substituting for f_1 from (3-35) into (4-6) and solving the resulting differential equation we obtain

$$\theta = \frac{1}{\frac{16}{9} R_e\,(a_2 + \beta b_1)\hbar(\hbar^2 - 1)} . \ln\left[\left(\frac{1+x}{1-x}\right)^{\hbar} . \left(\frac{\hbar - x}{\hbar + x}\right)\right] \qquad \text{..(4-7)}$$

where $\hbar = \left(\dfrac{a_1 + \beta b_3}{a_2 + \beta b_1} \right) < 0 \quad and \qquad \beta \in (-\infty, \infty) \setminus [-0.16, 0.044]$

and

$$\theta = \frac{-\hbar^2}{\frac{16}{9} R_e (a_2 + \beta b_1) \hbar (\hbar^2 + 1)} \left[\ln \left(\frac{1+x}{1-x} \right)^{\hbar} + 2 \tan^{-1} \left(\frac{x}{\hbar} \right) \right] \quad ..(4\text{-}8)$$

where $\hbar = \left(\dfrac{a_1 + \beta b_3}{a_2 + \beta b_1} \right) > 0 \quad and \qquad -0.16 \leq \beta < 0.044$

Here θ is measured from the point where the streamline cross the central plane $(x = 0)$. The (x, θ) relation is independent of the dimension of the cross-section.

For a given value of x, the range of θ varies with the dimensionless parameters R_e and β; in the case of Newtonian fluid $(\beta = 0)$ the range of θ varies inversely with R_e and for a fixed value of R_e the range of θ increase as β decreases. It is found that an increase in β leads to a decrease in the curvature of the streamlines in the central plane.

It is noted that the value of θ increases steadily with x and tends to infinity as x tends to unity and θ tends to minus infinity as x tends to minus one.

Numerical illustration are now given for a particular boundary and Reynolds number considered by Dean [8], namely $R_e = 63.3$, $\dfrac{d}{R} = \dfrac{1}{3}$ and for different values of the parameters β, k, L and time τ.

Fig.(7, 8), illustrate the streamline projection in the central plane. The streamline grows smoothly along the central plane and merges with the outer wall of the pipe. This shape is greatly affected by the non-linear stresses. The non-linear stresses force the flow to be around the inner wall for a quite angular distance, the flow centrifugal force forces the direction to sharply move in a radial direction but the flow steers near the outer wall again. This phenomenon becomes very clear as β, the non-Newtonian parameter, increase through the interval $(-\infty,\infty)\backslash$ [-0.16, 0.044], see Fig.(7). Inversely it is disappear as β varies from -0.16 to 0.044 Fig.(8)

4.2.2 Streamline Projection on the Cross-Section of the Pipe:

The streamline projection on the cross-section for a curved pipe are represented by

f_1 = Constant

Where f_1 is given by (3-20), which is combination of the radial and vertical velocity. The nature of the closed curved streamline for various fluid changes because of the non-Newtonian parameter.

The factors that affected on the secondary flow and θ-component velocity as can be seen from equations (3-26) and (3-27), are the frequency parameter k, the non-Newtonian parameter β, Dean number D and the time τ.

Sixty nine cases have been studied to cover the effect of each of these factors on the secondary flow and θ-component velocity. All figures (11-34) show that, there are two symmetrical regimes of secondary flow to appear in the cross-section in curved pipe. Also, it is noted that the intensity of the secondary flow is stronger in the middle of each of the upper and lower of the cross-section and becomes weaker when the more toward the boundary and the central plane.

For β increase through the interval $(-\infty, \infty)\backslash[-0.16, 0.044]$, $k = 1.77$ and $L = 0.01$ it is found that there is small vertical displacement away from the central plane, and the intensity of the secondary flow increases, see Fig.(11, 12).

In Fig.(13-16) when $\beta = 1$ and for k and L greater than zero, it is noted that the effect of k and L on the displacement of the secondary flow is the same as the effect of β and the intensity of secondary flow increase as k and L increase, but when β is small, e.g. $\beta = 0.044$ and different values of k and L, there is no displacement but there is change in intensity of the stream function, see Fig.(17-20).

Fig.(21-34) illustrated the effect of time on the streamline projection on the cross-section in curved pipe. In Fig.(21-28), the values of β, k and L are 1, 1.77, and 0.01 respectively and τ varies from 0 to 6.28. As τ varies from 0 to 2.05 (τ is measured in radian) there is displacement toward the central plane and the streamline become thicker near the central plane, see Fig.(21-23).

The transition stage from a two-vortex structure to a four-vertex structure occurs at $\tau = 2.061$; where two additional vortices start to grow near the corner of the inner and outer walls, see Fig.(24). They are clearer at $\tau = 2.07$, see Fig.(25) and the twin vortices rotating in opposite direction of the main vortices appear. Also, at τ increase it is noted that there are two stagnation regions near the corner of the inner and outer walls, Fig.(24), moving toward the center of the cross-section, Fig.(25). As τ increases, it is observed that the vortices in upper and lower half of cross-section near the corner of the inner and outer walls of the pipe expand and make another secondary flow, because of continuity displacement of the main vortices toward the central plane as τ increase, the new vortices control to the flow in pipe and become the main vortices, Fig.(26-28).

When the value of β is small, e. g. 0.044, and for the same values of k and L (i.e. $k = 1.77$ and $L = 0.01$), the increasing in τ from 0 to 6.28 lead to growth one vertex in each halve of the cross-section (upper and lower the central plane) near the boundaries, the vertices appear at $\tau \cong 1.753$, Fig.(30), and its direction opposite to main vortices. At τ varies from 0 to 6.28, the main vertices displace to the central plane. So it reach to stagnation regions, inversely the vertices that appear in upper and lower cross-section growth to take the location of the main vertices, see Fig.(29-34).

4.2.3 The Effect of Parameters, (β, k, L) and Time τ) on θ-Component Velocity:

The effect of parameters, (β, k, L and τ) on θ-component velocity illustrated in Fig.(35-47). It is noted that, parameters β, k and τ have weak effect on the location of center of axial velocity, and the increase in β and k leads to an increase on the value of the axial velocity. For increasing L there is horizontal displacement in the center of the axial velocity toward the outer wall of the pipe, see

Fig.(35-42). In Fig.(44-47) we noted that for small value for β, ($\beta = 0.044$), and the increase in k leads to increase in the intensity of the axial velocity but the increase in β and L have not effected, see Fig.(45,46).

4.3 Streamline Projection for Case2

As in case1, the differential equations of the streamline are

$$\frac{dX}{U} = \frac{(R+x)d\theta}{V} = \frac{dZ}{W} \qquad \text{.... (4-9)}$$

The velocity components, *(U, V, W)* are to be obtained from equations (3-49) and (3-50).

Up to sufficient accuracy equation (4-9) may be written as

$$\frac{dX}{U} = \frac{R\,h^4\,d\theta}{V_o(\frac{d^2}{h_2} - x^2)(1-z^2)} = \frac{dZ}{W} \qquad \text{.... (4-10)}$$

Also the expressions here appear in dimensional form.

4.3.1 Streamline Projection in the Central Pane:

This section has the same properties in the previous section (4.2.1) and the differential equation of the streamline in the central plane is

$$\frac{dx}{U} = \frac{R\,h^2 d\theta}{V_o(d^2 - x^2)} \qquad \text{.... (4-11)}$$

In case2 equation (4-4) becomes

$$U = \frac{\nu\, u}{h} \qquad \text{.... (4-12)}$$

Using equations (4-12) and (2-25), we obtain

$$U = \frac{\nu L}{h}\frac{\partial f_1}{\partial z} \qquad at \quad z = 0 \qquad \text{.... (4-13)}$$

where $L = 2R_e^2 \left(\frac{h}{R}\right)$

Substituting equation (4-13) into equation (4-11), gives

$$\frac{dx_1}{d\theta} = \frac{-2R_e}{(\frac{d^2}{h^2} - x_1^2)} \left.\frac{\partial f_1}{\partial z}\right|_{z=0} \qquad \text{.... (4-14)}$$

Substituting for f_1 from (3-49) into (4-14) and solving the resulting differential equation gives

$$\theta = \frac{1}{4.\ R_e(a_2 + \beta b_1)\hbar\left(\hbar^2 - \frac{4}{9}\right)} \left[\frac{3}{2}\ln\left(\frac{2/3 + x}{2/3 - x}\right)^{\hbar} + \ln\left(\frac{\hbar - x}{\hbar + x}\right)\right] \text{..(4-15)}$$

where $\hbar = \left(\frac{a_1 + \beta b_3}{a_2 + \beta b_1}\right) < 0 \quad and \qquad \beta \in (-\infty, \infty) \setminus [-0.64, 0.025]$

and

$$\theta = \frac{-\hbar^2}{4\ \ R_e(a_2 + \beta b_1)\hbar\left(\hbar^2 + \frac{4}{9}\right)} \left[\frac{3}{2}\ln\left(\frac{2/3 + x}{2/3 - x}\right)^{\hbar} + 2\ \tan^{-1}\left(\frac{x}{\hbar}\right)\right] \quad \text{..(4-16)}$$

where $\hbar = \left(\frac{a_1 + \beta b_3}{a_2 + \beta b_1}\right) > 0 \quad and \qquad -0.64 \le \beta < 0.025$

It noted that θ has the same properties as in section (4.2.1), but it tends to infinity as x tends to $2/3$ and it is tend to minus infinity as x tends to $-2/3$.

From Fig.(9, 10), we noted that the stream line projection in the center plane has the same phenomenon describe in section (4.2.1) associated with similar effect of β but in slowly form.

4.3.2 Streamline Projection on the Cross-Section of the pipe:

Figures (48-64) illustrate the effect of β, k, L and τ on the stream line projection on the cross-section in a curved pipe. It is found that there is no displacement in a secondary flow as β, k and L increase.

In addition, it is found that the intensity of the secondary flow increases as β, k and L increase, see Fig.(48-53). Also, it is noted that there are two stagnation regions near the inner and outer walls moving toward the center of cross-section as β, k and L increase.

As τ increases and the values of β, k and L are 10, 1.77 and 0.01 respectively, there is displacement toward the boundaries and the streamlines become thicker near the boundaries, Fig.(54). The transition stage from a stage from a two-vortex structure to a four-vertex structure occurs at $\tau = 1.85$; where two additional vortices start to grow near the inner and outer walls, see Fig.(55), the twin vortices rotating in opposite direction of the main vortices appear. Also, at τ increase it is noted that there are two stagnation regions near the inner and outer walls moving toward the center of the cross-section, see Fig.(54).

For $\tau > 1.85$, the stagnation regions start to move toward the center of cross-section causes displacement to main vortices toward the boundaries with the new vortices near the inner and outer walls move toward the center of cross-section to reach the main vortices, see Fig.(56-58).

Fig.(59-64), illustrate the effect of k, L and τ when β is small such as $\beta = 0.024$, it is noted that there is small displacement toward the central plane as β, k, L and τ increases and the intensity increase as these factors increase.

Finally, it is observed that the effect of each of the factors (β, k, L and τ) on θ-component velocity have the same effect in case1 (except L has stronger effected than in case1) see Fig.(65-79).

For steady state (time derivative is zero), in both cases, it noted that the effect of β and L have same effect as in unsteady state but in different level see Fig.(80-113).

Fig.(43, 73) explain the Newtonian type of fluid.

4.4 Comparison between Case1 and Case2 and Conclusion

For streamline projection in a central plane of the pipe, it is noted that as β increases, the effect in case1 is stronger than case 2.

Regarding streamline projection in the cross-section, in case1 it is noted that the increase in β, k and L lead to a weak displacement away from center plane and the intensity increases as these factors increase, where in case2, the increase in these factors lead to increase in the intensity (different from that in case1) of the secondary flow but there is no displacement.

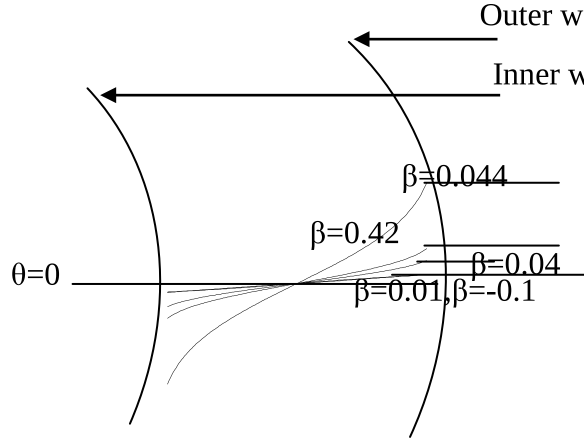
In addition to that, in case1 the increase in τ leads to a displacement toward the central plane and the streamline become thicker near central plane.

At $\tau = 2.061$ there exist four-vortex structure near the corner of the inner and outer walls of the pipe; while in case 2, the displacement was toward the boundaries occur and the streamline become thicker near the boundaries as τ increase. The four-vortex structure near the inner and outer wall appear at $\tau = 1.85$. Also, for small values of β, in case1 it is noted that there exist two-vortex structure and the displacement toward the central plane but there is no such they in case2.

4.5 Further Study

In what follow we give some suggestions for further study

1- The pressure in our problem is imposed. One can calculate the pressure by solving Possion's equation for pressure.
2- This work can be extended for helical pipe in which torsion is not equal to zero (in our problem torsion is zero).
3- This work can be extended for pipe with varying curvature.
4- Our problem can be resolving by using boundary layer method.



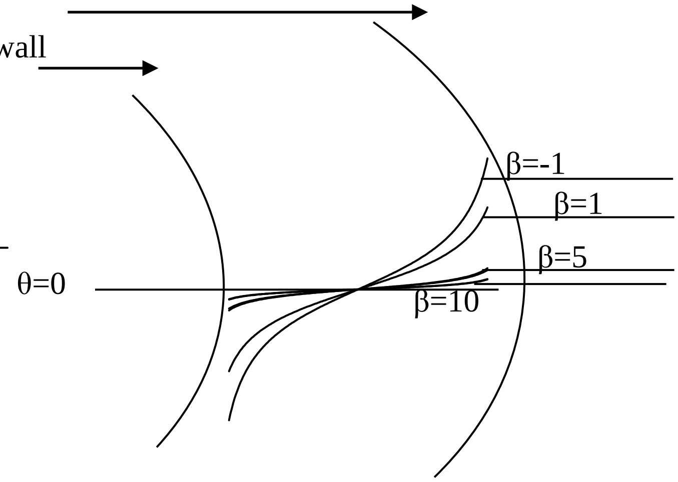
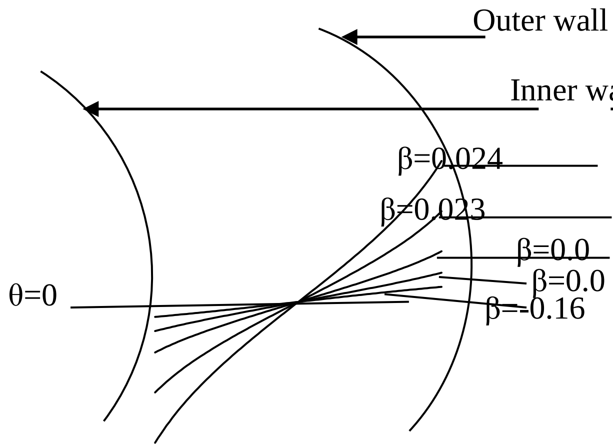


Fig.(7),streamline projection in the central
in the central
plane for β =-.1, 0.01, 0.04, 0.042, 0.044

Fig.(8),streamline projection
plane for β =-1, 1, 5, 10



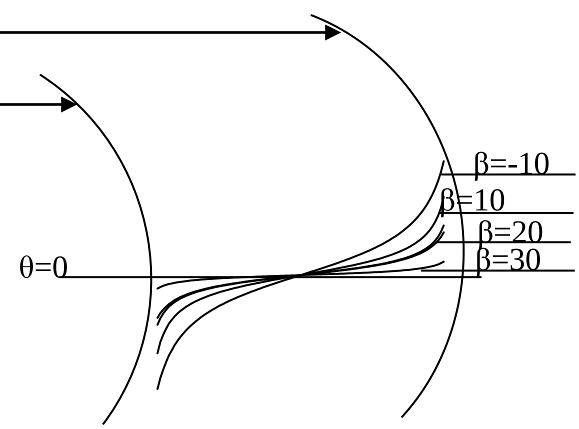


Fig.(9),streamline projection in the central
in the plane for β =-.16, 0.01, 0.02, 0.023, 0.024
10, 10, 20, 30

Fig.(10),streamline projection
central plane for β =-